The Arm Factorization

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Abstract
We construct the equivalent of the Taylor formula in the basis of all roots $\{x-k\}_K$ when K is $\mathbb{Z}\oplus i\mathbb{Z}$, $\mathbb{Q}\oplus i\mathbb{Q}$ and $\mathbb{C}$.

Introduction

The Taylor formula is the decomposition of a vector (a function) on a the monomial basis. Again the Fourier theory is the decomposition of a vector (a periodic function) on the complex exponential basis. In this case why a polynom couldn't be decomposed on the base $\{(x-k)\}_{k\in\mathbb{C}}$? Getting this decomposition is the main goal of this paper.

To get a decomposition on a basis, we first need to find the corresponding scalar (or inner) product of this space. To find this scalar product, we have to create a Kronecker delta. I solved this question in building this Kronecker between a discret variable k and a root r_p of the considered polynom $Z(z)$:

$$\delta_{k,r_p} = \lim_{z\to k} \frac{z-k}{z-r_p} \tag{0.1}$$

which is always zero except when k is r_p. Inspiring the Taylor formula, we identify

$$\lim_{z\to k} \frac{(z-k)}{(z-r_p)^{m_p}} = \lim_{z\to k} (z-k)\sum_{p=1}^{l} \frac{1}{(z-r_p)^{m_p}} = \lim_{z\to k} (z-k)\frac{\partial}{\partial x}\ln(Z(z)) \tag{0.2}$$

where m_p is the multiplicity of the root r_p and $\sum_{p=1}^{l} m_p = n = \deg(Z)$. The basic $\mathbb{Z}$-Arm factorization is this decomposition given in (1.6) for polynoms Z with roots in $\mathbb{Z}\oplus i\mathbb{Z}$.

Next I remarked that, if a polynom $Q(z)$ has its roots in $\mathbb{Q}$ and if we succed to find the common denominator q of all roots, then the polynom $P(\frac{x}{q})$ would have roots in $\mathbb{Z}$. This mysterious common denominator q can be seen in the summation of roots σ_1 (the coefficient before x^{n-1} modulo a constant). This idea leads to the $\mathbb{Q}$-Arm factorization which is the decomposition given in (3.17) for polynom with roots in $\mathbb{Q}\oplus i\mathbb{Q}$.

Furthermore, I searched for a final decomposition for all polynoms $C(z)$ i.e. with roots in $\mathbb{C}$. The problem is that we can't inspire of which is done before because roots are only points on the real axis and we need other tools for this decomposition. The tools I used to sum is of course the integration on the complex plane and that I used to make the integral nonzero is naturally the Dirac distribution $\delta(C(z))$ to give the roots modulo the multiplication by a derivate. Unhappynessly, this decomposition doesn't work for polynoms with nonsimple roots. I also introduced in the Remark 3 the notion of the Dirac distribution of a complex variable which is no well defined yet so the $\mathbb{C}$-Arm factorization doesn't work yet for complex. Thus the $\mathbb{C}$-Arm factorization is the application of those ideas for all polynom $C(z)$ with roots in $\mathbb{C}$.

In the first section, we introduce the $\mathbb{Z}$-Arm decomposition for polynom $Z(z)$ with roots in $\mathbb{Z}\oplus i\mathbb{Z}$. This decomposition gives the logarithm of the polynom, so we give in a corollary the exponential of the $\mathbb{Z}$-Arm factorization. In the second section, we illustrate this decomposition with the example of :

$$Z(z) = 2z(z-1+3i)(z-2-4i) \tag{0.3}$$

In the third section, we give the $\mathbb{Q}$-Arm factorization of a polynom $Q(z)$ with roots in $\mathbb{Q} \oplus i\mathbb{Q}$ and its exponential. In the fourth section, we apply it to the example of

$$Q(z) = 2\left(z - \frac{1}{2} - \frac{1}{3}i\right)\left(z - \frac{2}{3} - \frac{1}{2}i\right) \tag{0.4}$$

Also in the fifth section, we give the $\mathbb{C}$-Arm factorization of a polynom with simple roots in $\mathbb{C}$ and we apply it in the sixth section to the example of real roots :

$$C(x) = 3(x - e)(x - \pi^2) \tag{0.5}$$

1 $\mathbb{Z}$-Arm Factorization

Let $Z(z)$ be a polynom with zeros in $\mathbb{Z} \oplus i\mathbb{Z}$ i.e. $Z(z) = 0 \Longleftrightarrow z \in \mathbb{Z} \oplus i\mathbb{Z}$ where $i = \sqrt{-1}$. Then we have the following result

Theorem 1. *The $\mathbb{Z}$-Arm factorization of $Z(z)$ is given by :*

$$\ln(Z(z)) = \ln(\alpha) + \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \left[\lim_{z \to k+ik'} (z-k-ik') \ \frac{\partial}{\partial z} \ln(Z(x)) \right] \ln(z-k-ik') \tag{1.6}$$

where the constant α is given by : $\alpha = \frac{1}{n!} \ \frac{\partial^n Z(z)}{\partial z^n}$ and n is $n = \deg(Z(z))$

Proof :

Let

$$\left\{ r_1 + ir'_1, r_2 + ir'_2, ..., r_n + ir'_n \right\} \tag{1.7}$$

be all the zeros of the polynom $Z(z)$ such that $\forall p, r_p, r'_p \in \mathbb{Z}$. Then we can show that the coefficients of (1.6) are a scalar product between the elements $\{e_{k,k'} = (z-k-ik')\}_{k,k' \in \mathbb{Z}}$ of the basis of the space $(z - \mathbb{Z} - i\mathbb{Z})$ i.e. $< e_{k,k'}, e_{r_p, r'_p} >= \delta_{k,r_p}\delta_{k',r'_p}$:

$$\begin{aligned} < e_{k,j}, e_{r_p,r'_p} > &= \lim_{z \to k+ik'} (z-k-ik') \frac{\partial}{\partial z} \ln(z - r_p - ir'_p) \\ &= \lim_{z \to k+ik'} \frac{(z-k-ik')}{(z-r_p-ir'_p)} \\ < e_{k,j}, e_{r_p,r'_p} > &= \delta_{k,r_p}\delta_{k',r'_p} \end{aligned} \tag{1.8}$$

Next, we can factorize $Z(z)$ and write it with its zeros (1.7) :

$$Z(z) = \alpha \ \prod_{p=1}^{n} \ \left(z - r_p - ir'_p \right) \tag{1.9}$$

where α is the coefficient before z^n. It's trivial to show that we can obtain this coefficient is :

$$\alpha = \frac{1}{n!} \ \frac{\partial^n Z(z)}{\partial z^n} \tag{1.10}$$

So, given (1.8), we can write $Z(z)$ as :

$$\begin{aligned}
Z(z) &= \alpha \prod_{p=1}^{n} \left(z - r_p - ir'_p\right) \\
&= \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} \prod_{p=1}^{n} (z-k-ik')^{\delta_{r_p,k}\delta_{r'_p,k'}} \\
Z(z) &= \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} \prod_{p=1}^{n} (z-k-ik')^{\lim_{z\to k+ik'}(z-k-ik')\frac{\partial}{\partial z}\ln(z-r_p-ir'_p)} \\
Z(z) &= \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} (z-k-ik')^{\lim_{z\to k+ik'}(z-k-ik')\frac{\partial}{\partial z}\sum_{p=1}^{n}\ln(z-r_p-ir'_p)} \\
Z(z) &= \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} (z-k-ik')^{\lim_{z\to k+ik'}(z-k-ik')\frac{\partial}{\partial z}\ln(Z(z))} \qquad (1.11)
\end{aligned}$$

As a result, we can take the logarithm of (1.11) and we get the formula (1.6).

♦

We can reformulate the formula (1.6) in

Corollary 1. *For each polynom with zero in $\mathbb{Z} \oplus i\mathbb{Z}$, we have the decomposition*

$$Z(z) = \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} (z-k-ik')^{\lim_{z\to k+ik'}(z-k-ik')\frac{\partial}{\partial z}\ln(Z(z))} \qquad (1.12)$$

where α is given in Theorem 1.

Proof : See the proof of Theorem 1

♦

Remark 1. *It is evident that a polynom with zeros in $\mathbb{Z}$ instead of $\mathbb{Z} \oplus i\mathbb{Z}$ can be decompose with (1.6) or (1.12), it correspond to the case of $k' = 0$*

Remark 2. *The logarithm of a complex variable is well defined :*

$$\ln(z) = \ln(|z|) + i\arg(z) \qquad (1.13)$$

where $|z|$ is the modulus of z and $arg(z)$ is its argument.

2 Example Of $\mathbb{Z}$-Arm Factorization

We now take the example of

$$Z(z) = (-72 - 104i)z + (32 + 24i)z^2 - (10 + 10i)z^3 + 2z^4 \qquad (2.14)$$

We can see that the degree of $Z(z)$ is $n = 4$ and the coefficient before z^n is $\alpha = 2$. Then applying (1.6), we have the decomposition :

$$\begin{aligned}
\ln(Z(z)) &= \ln(2) + \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \left[\lim_{z \to k+ik'} (z-k-ik') \ \frac{\partial}{\partial z} \ln(Z(z)) \right] \ln(z-k-ik') \\
&= \ln(2) + \left[\lim_{z \to 0} x \ \frac{\partial}{\partial z} \ln\left((-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4 \right) \right] \ln(z) \\
&\quad + \left[\lim_{z \to 1-3i} (z-1+3i) \ \frac{\partial}{\partial z} \ln\left((-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4 \right) \right] \ln(z-1+3i) \\
&\quad + \left[\lim_{z \to 2+4i} (z-2-4i) \ \frac{\partial}{\partial z} \ln\left((-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4 \right) \right] \ln(z-2-4i) \\
&= \ln(2) + \left[\lim_{z \to 0} z \ \frac{(-72-104i) + (64+48i)z - (30+30i)z^2 + 8z^3}{(-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4} \right] \ln(z) \\
&\quad + \left[\lim_{z \to 1-3i} (z-1+3i) \ \frac{(-72-104i) + (64+48i)z - (30+30i)z^2 + 8z^3}{(-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4} \right] \ln(z-1+3i) \\
&\quad + \left[\lim_{z \to 2+4i} (z-2-4i) \ \frac{(-72-104i) + (64+48i)z - (30+30i)z^2 + 8x^3}{(-72-104i)z + (32+24i)z^2 - (10+10i)z^3 + 2z^4} \right] \ln(z-2-4i) \\
\ln(Z(z)) &= \ln(2) + 2\ln(z-2-4i) + \ln(z-1+3i) + \ln(z) \qquad (2.15)
\end{aligned}$$

So we have, in taking the exponential of each parts :

$$Z(z) = 2z(z-1+3i)(z-2-4i)^2 \qquad (2.16)$$

Maybe this example is a little bit difficult concerning the calculation. Of course, I programmed an algorithm on mathematica which did the calculation for me. Besides, I already knew what the zeros of $Z(z)$ were. If the reader wants, he can take a more simple example like $Z(z) = (z-1)(z-2)$ but I took this example to show it works with multiplicity 2 and with complex integer.

3 $\mathbb{Q}$-Arm Factorization

Now I generalize the $\mathbb{Z}$-Arm Factorization for polynom which have zeros in $\mathbb{Q} \oplus i\mathbb{Q}$ and I call it the $\mathbb{Z}$-Arm Factorization. Let $Q(z)$ be a polynom with zeros in $\mathbb{Q} \oplus i\mathbb{Q}$ i.e. $Q(z) = 0 \iff z \in \mathbb{Q} \oplus i\mathbb{Q}$ where $i = \sqrt{-1}$. Then we have the following result

Theorem 2. *The $\mathbb{Q}$-Arm factorization of $Q(z)$ is given by :*

$$\ln(Q(z)) = \ln(\alpha) + \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \left[\lim_{z \to \frac{k+ik'}{q}} \left(z - \frac{k+ik'}{q} \right) \frac{\partial}{\partial z} \ln(Q(z)) \right] \ln\left(z - \frac{k+ik'}{q} \right) \qquad (3.17)$$

where $\alpha = \frac{1}{n!}\frac{\partial^n Q(z)}{\partial z^n}$ *and q is the denominator of $\sigma_1 \in \mathbb{Q}$ (the coefficient before z^{n-1} of $\frac{Q(z)}{\alpha}$ if* $n = \deg(Q(z))$*).*

Proof :

As we do for the proof of Theorem 1, we write the zeros of the polynom $Q(z)$:

$$\left\{\frac{r_1}{q_1}+i\frac{r'_1}{q'_1},....,\frac{r_n}{q_n}+i\frac{r'_n}{q'_n}\right\} \tag{3.18}$$

with $\forall p, r_p, r'_p, q_p, q'_p \in \mathbb{Z}$. Then we can write $Q(z)$ as :

$$Q(z) = \alpha \prod_{p=1}^{n} \left(z-\frac{r_p}{q_p}+i\frac{r'_p}{q'_p}\right) \tag{3.19}$$

Besides, we know that

$$\sigma_1 = \sum_{p=1}^{n} \frac{r_p}{q_p}+i\frac{r'_p}{q'_p} \tag{3.20}$$

If we denote $\frac{1}{q}=\gcd(\frac{r_1}{q_1},...,\frac{r_n}{q_n},\frac{r'_1}{q'_1},...,\frac{r'_n}{q'_n})$ then $\exists \alpha_p=\frac{q}{q_p}, \alpha'_p=\frac{q}{q'_p}\in\mathbb{Z}$ such that

$$\sigma_1 = \sum_{p=1}^{n} \frac{\alpha_p r_p+\alpha'_p r'_p}{q} \tag{3.21}$$

So we can rewrite $Q(z)$ in (3.19) as

$$Q(z) = \alpha \prod_{p=1}^{n} \left(z-\frac{\alpha_p r_p+\alpha'_p r'_p}{q}\right) \tag{3.22}$$

Now we can write that :

$$q^n \; Q\left(\frac{z}{q}\right) = \alpha \prod_{p=1}^{n} \left(z-\alpha_p r_p+\alpha'_p r'_p\right) \tag{3.23}$$

wich is a polynom with zeros in $\mathbb{Z}$. So we can use Theorem 1 to obtain its $\mathbb{Z}$-Arm fractorization :

$$\ln\left(q^n Q\left(\frac{z}{q}\right)\right) = \ln(\alpha)+\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\lim_{z\to k+ik'}(z-k-ik')\;\frac{\partial}{\partial z}\ln\left(q^n Q\left(\frac{z}{q}\right)\right)\right]\ln(z-k-ik') \tag{3.24}$$

where

$$\alpha = \frac{q^n}{n!}\frac{\partial^n Q(\frac{z}{q})}{dz^n} = \frac{1}{n!}\frac{\partial^n Q(z)}{dz^n} \tag{3.25}$$

We can rewrite (3.24) as

$$\ln(Q(z)) = \ln(\alpha)+\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\lim_{z\to k+ik'}(z-k-ik')\;\frac{\partial}{\partial z}\ln\left(q^n Q\left(\frac{z}{q}\right)\right)\right]\ln\left(z-\frac{k+ik'}{q}\right) \tag{3.26}$$

If you make the changing of variable $z'=\frac{z}{q}$ in the limit (3.26), you wil have the formula :

$$\ln(Q(z)) = \ln(\alpha)+\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\lim_{z\to \frac{k+ik'}{q}}\left(z-\frac{k+ik'}{q}\right)\;\frac{\partial}{\partial z}\ln(Q(z))\right]\ln\left(z-\frac{k+ik'}{q}\right) \tag{3.27}$$

which gives the $\mathbb{Q}$-Arm factorization (3.17) because $\ln(q^n)$ is constant.

♦

Now, we can reformulate the formula (3.17) in

Corollary 2. *For each polynom with zero in $\mathbb{Q} \oplus i\mathbb{Q}$, we have the decomposition*

$$Q(z) = \alpha \prod_{k=-\infty}^{\infty} \prod_{k'=-\infty}^{\infty} \left(z - \frac{k+ik'}{q}\right)^{\lim_{z \to \frac{k+ik'}{q}} (z - \frac{k+ik'}{q}) \ \frac{\partial}{\partial z} \ln(Q(z))} \tag{3.28}$$

where α, q and n are given in Theorem 2.

Proof :

Take the exponential of (3.17)

♦

4 Example Of $\mathbb{Q}$-Arm Factorization

We choose to take the example of :

$$\begin{aligned} Q(z) &= \frac{1}{3} + \frac{17}{18}i - \left(\frac{7}{3} + \frac{5}{3}i\right)z + 2z^2 \\ Q(z) &= \left(\frac{6+17i}{36} - \frac{7+5i}{6}z + z^2\right)2 \end{aligned} \tag{4.29}$$

Then we have that $\sigma_1 = -\frac{7+5i}{6}$ so we can see that $q = 6$ and $\alpha = 2$.

So with (3.17), we have :

$$\begin{aligned} \ln(Q(z)) &= \ln(\alpha) + \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \left[\lim_{z \to \frac{k+ik'}{q}} (z - \frac{k+ik'}{q}) \ \frac{\partial}{\partial z} \ln\left(q^n Q(z)\right)\right] \ln\left(z - \frac{k+ik'}{q}\right) \\ &= \ln(2) + \left[\lim_{z \to \frac{3+2i}{6}} (z - \frac{3+2i}{6}) \ \frac{\partial}{\partial z} \ln\left(6^2\left(\frac{1}{3} + \frac{17}{18}i - \left(\frac{7}{3} + \frac{5}{3}i\right)z + 2z^2\right)\right)\right] \ln\left(z - \frac{3+2i}{6}\right) \\ &\quad + \left[\lim_{z \to \frac{4+3i}{6}} (z - \frac{4+3i}{6}) \ \frac{\partial}{\partial z} \ln\left(6^2\left(\frac{1}{3} + \frac{17}{18}i - \left(\frac{7}{3} + \frac{5}{3}i\right)z + 2z^2\right)\right)\right] \ln\left(z - \frac{3+2i}{6}\right) \\ &= \ln(2) + \left[\lim_{z \to \frac{3+2i}{6}} (z - \frac{3+2i}{6}) \ \frac{-42 - 30i + 72z}{6 + 17i - (42+30i)z + 36z^2}\right] \ln\left(z - \frac{3+2i}{6}\right) \\ &\quad + \left[\lim_{z \to \frac{4+3i}{6}} (z - \frac{4+3i}{6}) \ \frac{-42 - 30i + 72z}{6 + 17i - (42+30i)z + 36z^2}\right] \ln\left(z - \frac{4+3i}{6}\right) \\ \ln(Q(z)) &= \ln(2) + \ln\left(z - \frac{4+3i}{6}\right) + \ln\left(z - \frac{3+2i}{6}\right) \end{aligned} \tag{4.30}$$

We can rewrite (4.30) as :

$$Q(z) = 2\left(z - \frac{1}{2} - \frac{1}{3}i\right)\left(z - \frac{2}{3} - \frac{1}{2}i\right) \tag{4.31}$$

Now, I try to generalize the $\mathbb{Z}$-Arm factorization in the $\mathbb{C}$-Arm factorization

5 $\mathbb{C}$-Arm Factorization

Let $C(z)$ be a polynom with simple roots in $\mathbb{C}$ i.e. $C(z) = 0 \Longleftrightarrow z \in \mathbb{C}$. Then we have the following result

Theorem 3. *The $\mathbb{C}$-Arm factorization of $C(z)$ is given by :*

$$\ln(C(z)) = \ln(\alpha) + \int_{\mathbb{C}} dk \;\; \delta(C(k)) \; \left|\frac{\partial C(k)}{\partial k}\right| \;\; \ln(z-k) \tag{5.32}$$

where $\alpha = \frac{1}{n!}\frac{\partial^n Q(z)}{\partial z^n}$, $n = \deg(C)$, $|\;\;|$ *the absolute value and* δ *is the Dirac distribution which can be define as*

$$\delta(k) \;\; = \;\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \;\; e^{ikx} \tag{5.33}$$

Proof :

Let

$$\Big\{ r_1, ..., r_n \Big\} \;\; \in \;\; \mathbb{C}^l \tag{5.34}$$

the collection of all roots of $C(z)$.
Because of the definition (5.33) of the Dirac distribution and the equality

$$\delta(C(k)) = \sum_{p=1}^{n} \delta(k - r_p) \left| \frac{\partial k}{\partial C(k)} \right| \tag{5.35}$$

we can write :

$$\begin{aligned} \ln(C(z)) \;\; &= \;\; \ln\left(\alpha \prod_{p=1}^{n} (z - r_p) \right) \\ &= \;\; \ln(\alpha) + \sum_{p=1}^{n} \;\; \ln(z - r_p) \\ &= \;\; \ln(\alpha) + \sum_{p=1}^{n} \;\; \int_{\mathbb{C}} dk \;\; \delta(k - r_p) \;\; \ln(z-k) \\ &= \;\; \ln(\alpha) + \int_{\mathbb{C}} dk \;\; \left(\sum_{p=1}^{n} \delta(k - r_p) \left| \frac{\partial k}{\partial C(k)} \right| \right) \; \left| \frac{\partial C(k)}{\partial k} \right| \;\; \ln(z-k) \\ \ln(C(z)) \;\; &= \;\; \ln(\alpha) + \int_{\mathbb{C}} dk \;\; \delta(C(k)) \; \left| \frac{\partial C(k)}{\partial k} \right| \;\; \ln(z-k) \end{aligned} \tag{5.36}$$

which is (5.32).

♦

Remark 3. *The definition of the Dirac distribution $\delta(k)$ (5.33) with $k \in \mathbb{C}$ is well defined since*

$$\begin{aligned}
\delta(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ e^{i(\ \mathrm{Re}(k)\ +\ i\mathrm{Im}(k)\ )x} \\
\delta(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ e^{i\ \mathrm{Re}(k)\ x} e^{-\mathrm{Im}(k)\ x}
\end{aligned} \tag{5.37}$$

Corollary 3. *Each polynom $C(z)$ is given by it's exponential $\mathbb{C}$-Arm Factorization*

$$C(z) = \alpha e^{\int_{\mathbb{C}} dk \ \ \delta(C(k)) \ \ |\frac{\partial C(k)}{\partial k}| \ \ \ln(z-k)} \tag{5.38}$$

where α, δ are given in Theorem 3.

6 Example of $\mathbb{C}$-Arm Factorization

Because mathematica don't know yet what is the dirac distribution of a compkex number, we decide to choose an example with reals roots.

Let

$$C(x) = 3x^2 + 3e\pi^2 + 3x(-e - \pi^2) \tag{6.39}$$

for $x \in \mathbb{R}$, we can see that the coefficient before $x^{n=2}$ is $\alpha = 3$.
Hence formula (5.32) give :

$$\begin{aligned}
\ln(C(x)) &= \ln(\alpha) + \int_{\mathbb{R}} dk \ \ \delta(C(k)) \left| \frac{\partial C(k)}{\partial k} \right| \ \ln(x-k) \\
&= \ln(3) + \int_{\mathbb{R}} dk \ \ \delta(3k^2 + 3e\pi^2 + 3k(-e - \pi^2)) \ \left| 6k + 3(-E - \pi^2) \right| \ \ln(x-k) \\
\ln(C(x)) &= \ln(3) + \ln(-e + x) + \ln(-\pi^2 + x)
\end{aligned} \tag{6.40}$$

If we take the exponential of (6.40), we obtain :

$$C(x) = 3(x - e)(x - \pi^2) \tag{6.41}$$

Discussion

I've tried the formula (5.32) for polynom with simple root, it works. But if we take a polynom with multiple roots there would be problems with $\delta((x-r_p)^{m_p})$ I don't know why. By constrast the $\mathbb{Z}$-Arm Factorization and the $\mathbb{Q}$ one work for roots with multiplicity greater than one.

Again the Dirac distribution of a complex variable is not well defined yet but I still tried to used it because it extend the $\mathbb{C}$-Arm factorization to complex roots. I hope in the future somebody will succed to used this formula for complex roots. But for now mathematica seems to not understand when I ask it the Dirac distribution of a complex variable.

When we look at the formula (3.17) of the $\mathbb{Q}$-Arm factorization, we remark that the quotient $\frac{k+ik'}{q}$, which is discret numbers, would be continue if we take the limit $q \to \infty$. So if we change $q' = qb$ and taking $q', b \to \infty$, we will obtain an integral on reals numbers. But the problem is that this integral will be zero because roots a polynom are only points of the real axis. This is why we need a Dirac delta to make the integral nonzero.
Concerning the equation (5.38), there is a exponential of an integral. Instead of this expression, we want to do the same thing as (3.28) : obtain a product on discret variable to recognize the polynom. But the problem is that there is an integral in the expression (5.38) and not a simple discret summation. Then why not construct a continue product summation ?

$$e^{\int_a^b f(x)\mathrm{d}x} = \lim_{n\to\infty} e^{\sum_{k=0}^{n} \frac{b-a}{n} f(\frac{k(b-a)}{n})} = \lim_{n\to\infty} \prod_{k=0}^{n} e^{\frac{b-a}{n} f(\frac{k(b-a)}{n})} \tag{6.42}$$

Maybe the starting point of a latter work...